Management & Marketing (2009) Vol.4, No.2, pp. 165-170.

A STATISTICAL DISTRIBUTION USEFUL IN PRODUCT LIFE-CYCLE MODELING

Emil PETRESCU
Petromond Group, Bucharest

Abstract. *Starting from some recent results presented by Isaic – Maniu and Vodă (2008) regarding the Product Life – Cycle (PLC), we propose here an alternative statistical distribution in order to describe all four phases of a product life span. This distribution is the so-called ALPHA distribution which was formerly used in reliability theory as time-to-failure distribution.*

Keywords: PLC (Product Life-Cycle), life span, statistical modeling, Alpha distribution.

1. Introduction

As it is well-known, there are two main ways to approach the idea of PLC – Product Life Cycle. One is that to consider this subject matter as entirely belonging to the marketing theories and the second one which regards the concept in a reliability framework (Kotler, 1998, Blischke & Murthy, 2000).

The classical view on PLC assumes that there exist four phases which cover the life span of a given product: introduction to a specified market (characterized by „anemic" sales), growth (sudden increase of sales), equilibrium (or maturity phase – defined by approximately constant sales) and decline (when the sales decrease dramatically).

Some authors consider that we could face also a fifth phase namely that of withdrawal or disappearance from the market of the underlying product (Rink & Swan, 1979).

From a reliability perspective, the PLC is considered from producer's and client's points of view.

The buyer (client) is interested mainly in the duration from the purchase of a given product to its discardation at the end of item's useful service: this end could be reached if the object fails without possibility of repair or becomes extremely obsolete and does not satisfy anymore, the initial stated performances.

From manufacturer's perspective, the PLC is viewed as the time from the initial design/conception of the item to its „commercial end" – that is the withdrawal from that market. Betz (1993) or earlier, Chandran and Lancioni (1981) divide this PLC into two subphases: prelaunch and postlaunch – the latter consisting in marketing studies and post sale service.

It is important to notice that there exists another perspective namely that from the reliability management viewpoint. Some standardized documents – such as SAE M – 110/1993 and US MIL – HDBK 259 or IEC 300 – 3 – 3 – propose five phases (virtual model/concept, design, production/manufacture, operation and maintenance – if necessary), upgrade (or conversion) and scrap. The producer takes part mainly in the first three stages.

2. A discussion on some models of PLC curves

The traditional marketing theory adopted in the beginning the famous logistic curve of the Belgian scientist Pierre François Verhulst (1804-1849) in order to describe the PLC behaviour:

$$Y(t) = \frac{A}{B + Ce^{-Dt}}, t \geq 0, \quad A, B, C, D > 0 \tag{1}$$

Where:

$e \approx 2.718$ (Euler's number);

A, B, C, D being parameters which individualizes a certain situation.

Unfortunately, (1) can describe only the first three stages of PLC in the classical approach. Since if $t \to +\infty$ we get a horizontal asymptote Y=A/B, then the decline phase is not encompassed (we should have $Y(t) \to 0$ when $t \to +\infty$).

Several other models have been proposed (Pope, 1993; Demeure, 1997). The idea was to have a curve with two inflection points one on the left and the other on the right side of the maximum point, in order to describe the sudden increase and the sudden decrease of sales.

For instance, Al. N. Pop (2000) has advanced the following curve PLC – modelling:

$$f(t) = kt^a \exp(-bt), \; t \geq 0, \; a, b, k > 0 \tag{2}$$

which unlike the Verhulst's model, it has a maximum provided by the first derivative $f'(t) = 0$ as $t_{\max} = a/b$ with $f(t_{\max}) = k(a/b)^a \exp(-a)$ and two inflection points given by $f''(t) = 0$, their t – coordinates being $t_{1,2} = \left(a \pm \sqrt{b}\right)/b$.

The importance of these two inflection points in the PLC – modeling is the following: they show the evolution of sales – from „slow motion" to a significant growth and then the continuous decrease in sales followed by a rapid slide to the t – axis, that is with almost zero sales.

Actually, the Pop's function (2) is of a Gamma type density if this form is conveniently reparametrized as Isaic-Maniu and Vodă did (2008, pp. 94):

$$g(t; a, b) = \left(\frac{a}{b}\right)^a \frac{t^{a-1}}{\Gamma(a)} \exp(-at/b), \; t \geq 0, \; a, b > 0 \tag{3}$$

Where:

$\Gamma(a) = \int_0^{\infty} u^{a-1} e^{-u} du$ is Euler's Gamma function.

The above mentioned authors proposed also to combine the logistic with a Gamma-type function in the following way: first branch will be a truncated logistic in a given point $t = t_0$ and the second one should be a truncated Gamma form in the same point, as it is shown in figure 1.

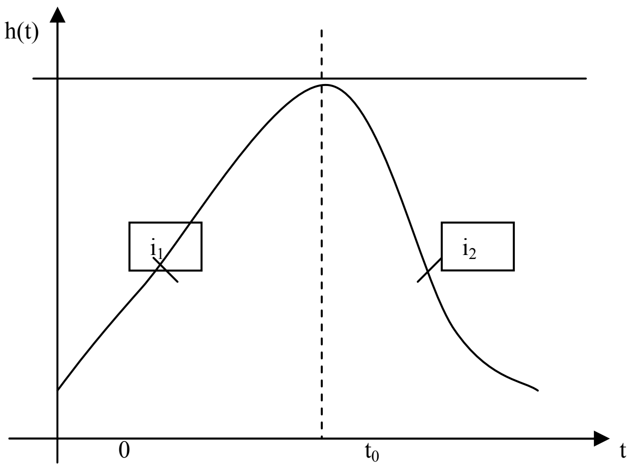

Figure 1. A combination beetween the logistic and a Gamma-type function.
(i_1 and i_2 are the corresponding inflection points)

Their approach seems logical since it describes the all four stages of the PLC, but raises some analitical problems, as follows:

1. one has to know the truncation point $t = t_0$;
2. the maximum of the new curve has to be situated under the horizontal asymptote of the logistic;
3. the new curve must have the form:

$$h(t) = p_1 Y(t) + p_2 g(t; a, b) \tag{4}$$

Where:

$p_1, p_2 > 0$, $p_1 + p_2 = 1$, $Y(t)$ and $g(t; a, b)$ are normalized in such a way to become probability densities, that is:

$$\int_0^{t_0} Y(t)dt = 1 \text{ and } \int_{t_0}^{\infty} g(t; a, b)dt = 1 \tag{5}$$

4. even if the proportions p_1 and p_2 are known, the parameter estimation for $h(t)$ is quite a difficult task (MLM – Maximum Likelihood Method or MM – Method of Moments require numerical procedures and do not give exact estimators)

Therefore, we shall propose a different approach advancing a distribution which was already used in applied statistics, but in a reliability context.

3. Alpha distribution as a model for PLC

An Alpha variable T is defined by the following density function:

$$T : f(t;\alpha,\beta) = \frac{\beta}{t^2\Phi(\alpha)\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\beta}{t}-\alpha\right)^2\right],\ x>0\ ,\ \alpha,\beta>0 \qquad (6)$$

Where:

$$\Phi(\alpha) \text{ is given as } \Phi(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} \exp(-u^2/2)du \qquad (7)$$

(the distribution function of the standard normal variate N(0,1) calculated for $t=\alpha$).

Introduced by Družinin (1977) to describe the time needed to perform some operations and applied by Katzev (1974) in cutting-tools durability analysis (Dorin & Vodă, 1973), this variable is in fact the inverse of a left truncated normal variable $N(\mu,\sigma^2)$ in the origin.

Indeed, if X is a measurable characteristic with $f(x;\theta)$ as its density and x_t is a left truncated point (that is $x \ge x_t$), then, the density of the truncated variable X_t is

$$X_t : f_t(x;\theta) = \frac{1}{1-F_X(x;\theta)} f(x;\theta)\ ,\ x \ge x_T\ ,\ \theta \in R \qquad (8)$$

where $F_X(x;\theta)$ is the distribution function of the initial variable X ($F_X^{'}(x;\theta) = f(x;\theta)$).

If we take $X \sim N(\mu,\sigma^2)$, $x_t = 0$ and compute the density of X_0^{-1}, denoting $\mu = \alpha/\beta$ and $\sigma = 1/\beta$ we get easily the form (6) – (Dorin et al. 1994, pp. 111-112).

Družinin advocates that in most of the cases, $\alpha \ge 3$ and since $\Phi(3) \approx 1$ (Militaru et al., 2000), we get the final form of Alpha density as:

$$T : f(t;\alpha,\beta) = \frac{\beta}{t^2\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\beta}{t}-\alpha\right)^2\right] \qquad (9)$$

$x>0\ ,\alpha \ge 3\ ,\ \beta>0$ (we have $\Phi(3) \approx 0.99865 \approx 1$).

This curve is fitted to modelize all four phases of PLC since it has two inflection points, a maximum of sales obtained for the model value provided by $f^{'}(t;\alpha,\beta)$ as

$$t_{mo} = \frac{\beta}{4}\left(\sqrt{\alpha^2+8}-\alpha\right) \tag{10}$$

and if t is large, the sales tend to zero (the graph is asymtptic to the t-axis).

The average amount of sales is given by the mean-value of T, that is

$$E(T) = \frac{\beta}{\alpha}\left(1+\frac{1}{\alpha^2}\right) \tag{11}$$

and the median amount of sales is $t_{me} = \beta/\alpha$.

It is interesting to observe that if α is large (for instance $\alpha = 10$), the average sales are very close to the median ones. Notice also that the ratio between the modal value and the median is

$$\frac{t_{mo}}{t_{me}} = \frac{\alpha}{4}\left(\sqrt{\alpha^2+8}-\alpha\right)$$

and this relationship could provide easily an estimate for α, taking into account the empirical values of the mode and the median. The second parameter β is then estimated as $\hat{\beta} = \hat{t}_{me}\hat{\alpha}$.

We draw the conclusion that our proposal is a lot easier than that of Isaic-Maniu and Vodă discussed above.

Some other advantage is the following: considering the first inflection point of the curve (see figure 2).

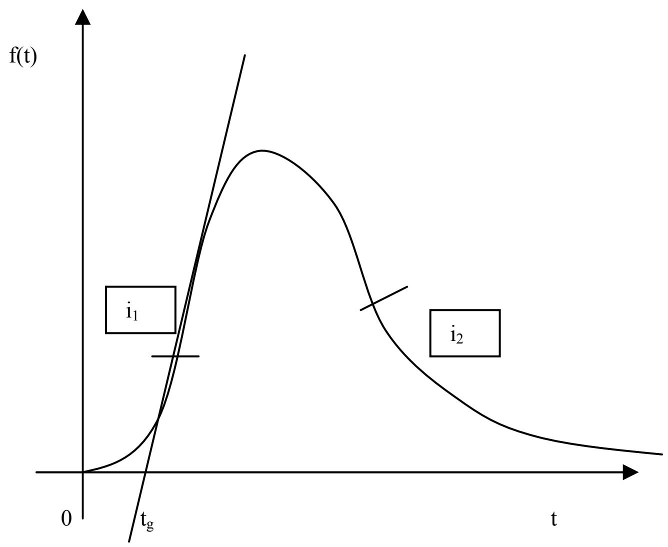

Figure 2. First inflection point as sale growth predictor

We can predict the moment wherefrom the sales will grow rapidly by drawing the tangent in i_1: the intersection of this line with t-axis will be the value t_g which could be interpreted as the starting moment of a rapid growth in sales.

The ideea to transform the PLC-curve (such as Pop's one or even the logistic) in a probability density is fruitful since one may compute in this way relevant statistical indicators such average/median sales, modal sales a.s.o. On the other hand one may evaluate also the probability that the actual sales are less (or more) than a predicted value.

References

Betz, F. (1993), *Strategic Technology Management*, New York: McGraw-Hill Book Co.

Blischke, W. R., Murthy, D. N. P. (2000), *Reliability. Modeling, Prediction and Optimization*, New York: John Wiley & Sons Inc.

Chandran, R., Lancioni, R. A. (1981), Product recall: A challenge for the 1980's *International Journal of Physical Distribution and Materials Management*, **11**, pp. 46-55

Demeure, Cl. (1997), *Marketing*, Paris: Dalloz Editeurs

Dorin, Al. C., Vodă, V. Gh. (1973), Fiabilitatea sculelor aşchietoare, *Calitatea Producției şi Metrologiei*, Anul III, nr. 11, pp. 650-655

Dorin, Al. C., Isaic-Maniu, Al., Vodă, V. Gh. (1994), *Probleme statistice ale fiabilității*, Bucureşti: Editura Economică

Družinin, G. V. (1977), *Nadežnosti System Avtomatiki (izdanie vtoroye dopolnitelnoe)*, Moskva: Izdatelstvo ENERGHIA

Isaic-Maniu, Al., Vodă, V. Gh. (2008), On a model regarding the product life-cycle, *Management & Marketing*, **3**, no. 3(11) Autumn/2008, pp. 87-96

Katzev, P. G. (1974), *Statisticheskye Metody Issledovanye Režushchego Instrumenta*, Moskva: Izdatelstvo Mashnostroenie

Kotler, Ph. (1998), Managementul Marketingului, Bucureşti: Editura Teora

Militaru, C., Petrescu, E., Dumitrescu, A. (2000), *Statistică Teoretică*, Bucureşti: Editura Printech

Pope, J. L. (1993), *Practical Marketing Research (New Updated edition)*, New York: AMA-American Marketing Association

Rink, D. R., Swan, J. E. (1979), Product life cycle research: A literature review, *Journal of Business Research*, no. 78, pp. 219-242

***SAE M-110 (1993), *Reliability and Maintainability for Manufactoring Machinery and Equipment*, Society of Automotive Engineers. Inc.

***US MIL – HDBK 259, *Life Cycle Costs in Navy Acquisitions*

***IEC 300-3-3, *Life Cycle Costing*